Published by Institute of Physics Publishing for SISSA/ISAS

Received: *June 20, 2002*
Accepted: *June 21, 2002*

Strings in a time-dependent orbifold

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Abstract: We consider string theory in a time dependent orbifold with a null singularity. The singularity separates a contracting universe from an expanding universe, thus constituting a big crunch followed by a big bang. We quantize the theory both in light-cone gauge and covariantly. We also compute some tree and one loop amplitudes which exhibit interesting behavior near the singularity. Our results are compatible with the possibility that strings can pass through the singularity from the contracting to the expanding universe, but they also indicate the need for further study of certain divergent scattering amplitudes.

Keywords: Bosonic Strings, Superstrings and Heterotic Strings, Conformal Field Models in String Theory.

JHEP06(2002)045

Contents

1. Introduction and motivation

Much of the work in string theory and CFT has been in the context of *euclidean signature* target space, i.e. in *time-independent* background geometries. Many interesting problems in physics involve time in an essential way. It is therefore of interest to extend the study of string theory to time-dependent backgrounds.

The extension of string theory from euclidean to lorentzian signature is nontrivial and many new issues arise. For example, in a time-dependent background there is no natural definition of the vacuum. Also, it is not always clear what the correct observables of string theory are. Another question we may ask is whether time comes to an end. If it does, how do we describe the boundary conditions or final states. If it does not, how do strings resolve or pass through the spacelike singularities predicted by general relativity. Such singularities arise behind the horizon of black holes and in the big-bang. Therefore understanding them is of great interest. We may also ask whether timelike and null closed curves are pathological, and if not, whether there are interesting string-winding effects associated with such curves. The list of questions and issues goes on, but the above questions are representative.

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In this paper we begin an exploration of these questions in string theory by describing a simple model of a time-dependent geometry in which one can attempt to address some of the above issues in a controlled setting. In a companion longer paper [1] we will provide more details.

In section 2 we describe the model and its geometry. In section 3 we study the functions on our spacetime which are the wave functions of the first quantized particles. In sections 4 and 5 we quantize free strings in the light-cone and conformal gauges and compute the torus partition function. Section 6 is devoted to a preliminary analysis of the interactions and backreaction. Our conclusions are presented in section 7

2. Geometry of the orbifold

2.1 The model

Time dependent backgrounds are difficult to work with in general. As with Calabi-Yau compactification, orbifolds provide a useful approach: They are simple enough to be solvable, yet complicated enough to illustrate nontrivial effects. An interesting class of time-dependent models is based on target spaces of the form $(\mathbb{R}^{1,n}/\Gamma) \times \mathcal{C}^{\perp}$ where the orbifold group is a discrete subgroup Γ of the Poincaré group, and $\mathcal{C}^{\perp}$ is a "transverse" conformal field theory rendering the full string theory consistent.[1] This class of models was discussed about 12 years ago by Horowitz and Steif [2]. Recently there has been a renaissance in the subject motivated in part by the desire to use string theory to address questions of cosmology [3]–[8]. Other time dependent backgrounds were studied in [9]–[14].

The model studied in this letter is based on the target space $(\mathbb{R}^{1,2}/\Gamma) \times \mathcal{C}^{\perp}$ where $\Gamma \cong \mathbf{Z}$ is a subgroup of the 3D Lorentz group $\mathrm{Spin}(1,2) \cong \mathrm{SL}(2,R)$. The orbifold group Γ is completely specified by choosing a conjugacy class of a generator g_0. $\mathrm{SL}(2,R)$ has three distinct conjugacy classes, elliptic, hyperbolic, and parabolic. The elliptic classes correspond to spatial rotations, the hyperbolic classes correspond to boosts leaving one spatial dimension fixed, and the parabolic classes correspond to "null boosts." We will choose g_0 to be a parabolic element. This leads to the "null orbifold" introduced in [2], briefly studied in [15] and more recently considered in [7]. We will now describe the geometry of this orbifold in some detail.

We introduce coordinates x^μ on $\mathbb{R}^{1,2}$ which are assembled into a column vector X. The Lorentz metric is $ds^2 = -2dx^+dx^- + dx^2$. The generator g_0 acts as

$$X := \begin{pmatrix} x^+ \\ x \\ x^- \end{pmatrix} \to g_0 \cdot X = e^{v\mathcal{J}} X = \begin{pmatrix} x^+ \\ x + vx^+ \\ x^- + vx + \frac{1}{2}v^2 x^+ \end{pmatrix}; \qquad \mathcal{J} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.1}$$

That is, $g_0 = \exp(ivJ)$ where we take the Lie algebra generator

$$J = \frac{1}{\sqrt{2}}(J^{0x} + J^{1x}) \tag{2.2}$$

corresponding to a linear combination of a boost and a rotation. By a boost in the 1-direction we can set $v = 2\pi$.

[1] One can also consider the case where Γ also acts on "transverse" coordinates.

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The geodesic distance between a point and its n'th image is $|nvx^+|$. Therefore our orbifold has no closed timelike curves. For $x^+ \neq 0$ all closed curves are spacelike and for $x^+ = 0$ there exist closed null curves.

The orbifold by Γ breaks the Poincaré symmetry, leaving only two of its Lie algebra generators unbroken. These are J of (2.2) and $p^+ = -p_-$ which shifts x^- by a constant. The null Killing vector associated with p^+ allows us to pick light-cone gauge; i.e. to treat x^+ as time. Since $p^- = -p_+$ is broken by the orbifold the light-cone system depends on x^+ and hence it is still nontrivial. Having a null Killing vector has an important consequence. The light-cone evolution is first order in light-cone time and hence it is simpler than standard second order time evolution. As a result of that, even though our background is time dependent, there is no particle production in the second quantized theory when described in the light-cone frame.

Equation (2.1) determines the action of the group element g_0 on spinors up to a sign. For an appropriate choice of this sign the group Γ leaves one spinor invariant, and therefore the orbifold has a covariantly constant spinor. When the superstring is compactified on this orbifold it preserves half of the supercharges. These supercharges square to the Killing vector p^+.

In the light-cone frame where x^+ is taken to be the time $x^+ = \tau$ the transformation (2.1) has the following physical interpretation. It is simply a galilean boost by velocity v. It leaves the time $x^+ = \tau$ invariant and shifts the coordinate $x \to x + v\tau$. The action of the parabolic generator g_0 on the translation generators P is similar to (2.1)

$$P = \begin{pmatrix} p^+ \\ p \\ p^- \end{pmatrix} \to e^{v\mathcal{J}} P = \begin{pmatrix} p^+ \\ p + vp^+ \\ p^- + vp + \frac{1}{2}v^2 p^+ \end{pmatrix} . \tag{2.3}$$

Since $m^2 = 2p^+p^- - p^2$ is Lorentz invariant, it is invariant under the parabolic generator J. An analogy to newtonian physics emerges when we solve for $p^- = \frac{p^2+m^2}{2p^+}$. In the light-cone frame p^+ is interpreted as the mass $\mu = p^+$, $V = \frac{m^2}{2p^+}$ is the potential energy and $p^- = \frac{p^2}{2\mu}+V$ is the total energy. In terms of the variables μ, p, V the parabolic transformation (2.3) is simply $p \to p + v\mu$ with μ and V left invariant.

Thus the parabolic orbifold obtained from (2.1) can be considered in the light-cone frame as the quotient by a galilean boost.

2.2 A little model for a big bang

It is convenient describe the geometry of the quotient space $\mathcal{O} = \mathbb{R}^{1,2}/\Gamma$ by introducing new coordinates

$$\begin{aligned} y^+ &:= x^+ \\ y &:= \frac{x}{x^+} \\ y^- &:= \frac{2x^+x^- - x^2}{2x^+} = x^- - \frac{1}{2}\frac{x^2}{x^+} . \end{aligned} \tag{2.4}$$

The advantage of this change of coordinates is that the identifications are simple (henceforth we take $v = 2\pi$):

$$(y^+, y, y^-) \sim (y^+, y + 2\pi, y^-) \tag{2.5}$$

and so is the metric

$$ds^2 = -2dx^+dx^- + (dx)^2 = -2dy^+dy^- + (y^+)^2(dy)^2\,. \tag{2.6}$$

The spacetime (2.6), which we call the parabolic pinch, may be visualized as two cones (parametrized by y^+ and y) with a common tip at $y^+ = 0$, crossed with the real line (for y^-). y plays the role of an "angular variable" and the null coordinate y^+ plays the role of a "radial variable." As a function of the "light-cone time" y^+ we have a big crunch of the circle at $y^+ = 0$ which is followed by a big bang. The dual role of y^+ as both a radial variable and a time variable will be the source of some interesting physics.

It is important that the model does not have closed timelike curves. The closed loop parametrized by y is spacelike for $y^+ \neq 0$. At the singularity where $y^+ = 0$ the circumference of the y circle vanishes. We will discuss the singularity in more detail below.

Having in mind light-cone frame, we will refer to all the points in the orbifold with $x^+ = y^+ < 0$ as the past cone, and to all the points in the orbifold with $x^+ = y^+ > 0$ as the future cone. An interesting feature of the spacetime (2.6), which is intuitively what we expect from a big crunch followed by a big bang, is that every point $\mathcal{P} = (y^+, y, y^-)$ with $y^+ > 0$ in the future cone is in the causal future of every point $\widetilde{\mathcal{P}} = (\widetilde{y}^+, \widetilde{y}, \widetilde{y}^-)$ with $\widetilde{y}^+ < 0$ in the past cone. This follows since the lorentzian distance square between $\mathcal{P}$ and the n^{th} image $g_0^n\widetilde{\mathcal{P}}$ of $\widetilde{\mathcal{P}}$ can be computed from

$$||X - g_0^n\widetilde{X}||^2 = -2\Delta x^+\Delta x^- + (\Delta x)^2 + (2\pi n)^2 x^+\widetilde{x}^+ + 2(2\pi n)\left(x^+\widetilde{x} - x\widetilde{x}^+\right) \tag{2.7}$$

where $\Delta x^\mu = x^\mu - \widetilde{x}^\mu$. At large-$N$ the term $(2\pi n)^2 x^+\widetilde{x}^+$ dominates, so if $x^+\widetilde{x}^+ < 0$, there are infinitely many $g \in \Gamma$ such that $\mathcal{P} \in I^+(g \cdot \widetilde{\mathcal{P}})$.

We may now formulate our motivating questions more precisely in this context, namely:

1. What is the nature of the singularity at $x^+ = y^+ = 0$? What happens to string theory there?

2. Is the singularity an end of spacetime; i.e. does a consistent formulation of string theory on $\mathbb{R}^{1,2}/\Gamma$ require *one cone* or *two cones*? Or is this a choice of physical model?

2.3 Nature of the singularity

In this subsection we analyze the singular subspace $x^+ = y^+ = 0$. The identification on this subspace is

$$\begin{pmatrix} x^+ = 0 \\ x \\ x^- \end{pmatrix} \sim \begin{pmatrix} x^+ = 0 \\ x \\ x^- + 2\pi n x \end{pmatrix} \tag{2.8}$$

Unlike the situation for $x^+ \neq 0$, here x is not subject to identification and hence it is a good coordinate. Note also that there is no nontrivial identification at $x = 0$, so all points along the x^- axis are distinct. On the other hand, for very small x, points with very different x^-, with very small spacing, will be identitfied. This situation is impossible to describe using the y coordinates.

Since the coordinate transformation (2.4) is singular at $x^+ = 0$ we need to describe the space C_Y coordinatized by y with some care. (2.6) implies that we begin with $(y^+, y, y^-) \in \mathbb{R}^3$, and quotient by the equivalence relation $(y^+, y, y^-) \sim (y^+, y + 2\pi, y^-)$ together with $(y^+ = 0, y, y^-) \sim (y^+ = 0, 0, y^-)$. This latter identification is natural since y is an angular variable and y^+ behaves like a radial variable. More precisely, C_Y projects to 2-dimensional Minkowski space parametrized by (y^+, y^-) with generic fiber a circle, except at $y^+ = 0$, where the fiber degenerates to a point. C_Y has no closed timelike or null curves and is Hausdorff. It looks like a double cone times a line.

In the above we gave a precise definition of the space C_Y following from (2.6). This space is in fact not precisely the orbifold $\mathcal{O} = \mathbb{R}^{1,2}/\Gamma$ of section 2, but is closely related to it. Our change of coordinates is a continuous map

$$\pi : C_Y \to \mathcal{O} = \mathbb{R}^{1,2}/\Gamma \tag{2.9}$$

given explicitly by

$$\begin{aligned} x^+ &= y^+ \\ x &= y y^+ \\ x^- &= y^- + \frac{1}{2} y^+ y^2 \end{aligned} \tag{2.10}$$

This is an isomorphism for $x^+ = y^+ \neq 0$, but is not even surjective for $y^+ = 0$. In fact, the space $\mathcal{O}$ is not Hausdorff. Recall that the Hausdorff separation axiom states that open sets separate distinct points. That is, $\forall P \neq Q$, $\exists$ open sets U_P, U_Q containing P, Q, respectively, such that $U_P \cap U_Q = \emptyset$. To illustrate the non-Hausdorff nature of the orbifold consider the simplified problem of the quotient of the plane $x^+ = 0$ by the identification (2.8). As we have have mentioned, all points along the x^- axis ($x^+ = x = 0$) are distinct. On the other hand, for small x the open sets which resolve different values of x^- must also get small, and this leads to a non-Hausdorff topology. Specifically, one finds that points on the lines $L_A = \{(0, A, x^-) : x^- \in \mathbb{R}\}$ and $L_{-A} = \{(0, -A, x^-) : x^- \in \mathbb{R}\}$ cannot be separated by open sets in $\mathcal{O}$. Of course, by adding an equivalence relation to $\mathcal{O}$: $(x^+ = 0, x, x^-) \sim (x^+ = 0, x = 0, x^-)$ or by considering its subspace which is the image of (2.9) we produce a new space $\mathcal{O}'$ topologically isomorphic to C_Y.

To summarize: in studying our time-dependent string background we are lead to consider two distinct spaces, the group quotient $\mathcal{O}$, which is non-Hausdorff, and the parabolic pinch $\mathcal{O}'$, which is Hausdorff. They are identical away from the singularity but the nature of the singularity in the two spaces is different. One useful way of thinking about the distinction is in terms of the foliation by equal-time slices. $\mathcal{O}$ is foliated by slices $\mathcal{F}_{x^+}$, where $\mathcal{F}_0$ is not Hausdorff, while $\mathcal{O}'$ is foliated by slices $\mathcal{F}'_{y^+}$, where $\mathcal{F}'_0 = \{(0, 0, y^-)\}$ is a real line. For $x^+ = y^+ \neq 0$ the map (2.9) defines an isomorphism of the foliated spaces.

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The advantage of the y coordinate system is that it gives us a clear picture of the topology both of $\mathcal{O}$ and of $\mathcal{O}'$ for $x^+ = y^+ \neq 0$. On $\mathcal{O}'$, y^μ is a good global coordinate system including the singularity on the double cone times a line. On the group quotient $\mathcal{O}$ the y coordinate system is singular at $\mathcal{F}_0$. This is clear from the identification (2.8). Therefore, for questions associated with the singularity of $\mathcal{O}$ we will prefer to use the x coordinate system.

The standard string orbifold procedure constructs string theory on the quotient by a *group* action. In formulating strings in light-cone gauge it might be possible to construct strings propagating on $\mathcal{O}'$, as well as on $\mathcal{O}$. Unfortunately, the consistency of string theory on $\mathcal{O}'$ is not self-evident since $\mathcal{O}'$ is not geodesically complete at $y^+ = 0$; some geodesics reach $y^- \to \pm\infty$ in finite proper time. In the covariant formulation described below, we are describing strings propagating on $\mathcal{O}$.

2.4 Relation to other models

The parabolic orbifold is closely related to two other models, which, at first sight, appear to *require* that physics only makes sense on one cone, and not two. We believe that in each of these examples there is a subtlety which invalidates the argument for a single cone.

Our first example is the elliptic orbifold: $(C/\mathbf{Z}_N) \times \mathbb{R}$, boosted in the (r, t) plane by a boost $\sim 1/N$, in the limit as $N \to \infty$. Locally, the metric degenerates to that of C_Y. However, one must carefully consider the *range* of the coordinates. There are regions of spacetime mutually inaccessible in the initial and final systems: It is not even true that the one-cone theory is the limit of the elliptic orbifold.

Our second example is the $J = M = 0$ BTZ black hole [16, 17]. Let us identify $\mathbb{R}^{1,2}$ with the Lie algebra of $\mathrm{SL}(2, R)$. Then (2.1) is simply the adjoint action by g_0. Now, the BTZ black hole is simply obtained by replacing the Lie algebra by the Lie group $\widetilde{\mathrm{SL}(2, R)}$,[2] and by promoting the adjoint action of Γ on the Lie algebra to the adjoint action on the group, and restricting to a single fundamental domain. The Lie algebra is the infinitesimal region of the identity and indeed a scaling limit of the BTZ metric near the singularity produces (2.6). (In this context the fact that the singularity is not Hausdorff was noted in [17].) One might think that the AdS/CFT correspondence demands that we do not continue beyond the singularity. But careful consideration suggests that there is no immediate contradiction between the AdS/CFT correspondence and the existence of a possible continuation beyond the singularity.

3. First quantized theory

In this section, as a warmup for string theory on the orbifold, we consider first quantized particles. The first quantized wave equation for a spinless particle of mass m is:

$$\left[-2\frac{\partial}{\partial x^+}\frac{\partial}{\partial x^-} + (\frac{\partial}{\partial x})^2\right]\psi = m^2\psi\,. \tag{3.1}$$

[2] $\widetilde{\mathrm{SL}(2, R)}$ is the universal cover of $\mathrm{SL}(2, R)$.

To define the orbifold Hilbert space we project onto wavefunctions invariant under

$$\mathcal{U}(g_0) := \exp\left(2\pi i \widehat{J}\right), \qquad \widehat{J} = \widehat{x}^+\widehat{p} - \widehat{x}\widehat{p}^+ = -i\left(x^+\frac{\partial}{\partial x} + x\frac{\partial}{\partial x^-}\right). \tag{3.2}$$

The generators of the Poincare algebra which are invariant under $\mathcal{U}(g_0)$ are $\widehat{p}^+$ and $\widehat{J}$. Thus on the orbifold it is convenient to diagonalize these operators. Explicitly,[3]

$$\begin{aligned}\psi_{p^+,J} &= \sqrt{\frac{p^+}{ix^+}}\exp\left[-ip^+x^- - i\frac{m^2}{2p^+}x^+ + i\frac{p^+}{2x^+}(x-\xi)^2\right] \\ &= \int_{-\infty}^{\infty}\frac{dp}{\sqrt{2\pi}}\, e^{-ip\xi}\phi_{p^+,p}(x^+,x^-,x)\end{aligned} \tag{3.3}$$

where $\xi := -J/p^+$ and hence the eigenvalue J enters as a "position" ξ. In the second line above we have expanded the wave function in terms of standard on-shell plane wave basis

$$\phi_{p^+,p}(x^+,x,x^-) = \exp\left(-ip^+x^- - ip^-x^+ + ipx\right), \qquad p^- = \frac{p^2+m^2}{2p^+} \tag{3.4}$$

The plane waves are not invariant under the orbifold action (2.1), and the invariant functions obtained by summing over the images are not a convenient basis on the orbifold. It is easy to check that in terms of $\psi_{p^+,J}$ (3.3) the orbifold projection is simply $J \in \mathbf{Z}$.

Since $\psi_{p^+,J}$ is a Fourier transformation in p it can be interpreted as an x-eigenfunction. Indeed

$$\lim_{x^+\to 0}\psi_{p^+,J}(x^+,x,x^-) = \sqrt{2\pi}e^{-ip^+x^-}\delta(x-\xi), \qquad \xi = -J/p^+ \tag{3.5}$$

This result can be derived more directly by considering an eigenvector of ∂_{x^-} which is well defined on the surface $\mathcal{F}_0$ under the identification (2.8).[4] Equation (3.5) can also be understood from the fact that $\widehat{J} = \widehat{x}^+\widehat{p} - \widehat{x}\widehat{p}^+ \to -\widehat{x}\widehat{p}^+$ at $x^+ = 0$. Thus the eigenfunction of $\widehat{J}$ and $\widehat{p}^+$ must be a coordinate eigenfunction with eigenvalue $\xi = -J/p^+$. Since J is quantized on the orbifold, we see that with fixed p^+ the wavefunctions are supported on the lattice $x \in \frac{1}{p^+}\mathbf{Z}$ at $x^+ = 0$.

Equation (3.5) should be interpreted with care. The limit of ψ is a distribution and should only be convoluted with smooth functions. For example, it is clear that $\lim_{x^+\to 0}|\psi_{p^+,J}|^2 = |p^+/x^+|$ which is not localized at ξ.

4. Strings on the orbifold: light-cone gauge

In this section we analyze the system in the light-cone gauge $x^+ = y^+ = \tau$. The light-cone gauge lagrangian is

$$\begin{aligned}L &= -p^+\partial_\tau x_0^- + \frac{1}{4\pi\alpha'}\int_0^{2\pi} d\sigma\left(\alpha' p^+\partial_\tau x\partial_\tau x - \frac{1}{\alpha' p^+}\partial_\sigma x\partial_\sigma x\right) \\ &= -p^+\partial_\tau y_0^- + \frac{1}{4\pi\alpha'}\int_0^{2\pi} d\sigma\tau^2\left(\alpha' p^+\partial_\tau y\partial_\tau y - \frac{1}{\alpha' p^+}\partial_\sigma y\partial_\sigma y\right)\end{aligned} \tag{4.1}$$

[3]We will take $p^+ \neq 0$. Some further comments on the $p^+ = 0$ case will be found in [1].

[4]Note that the limit (3.5) is not well defined on the Hausdorff space $\mathcal{F}_0' \subset \mathcal{O}'$ since the distribution (3.5) separates points which are identified in $\mathcal{O}'$.

where

$$\begin{aligned} x(\sigma,\tau) &= \tau y(\sigma,\tau) \\ y_0^-(\tau) &= x_0^-(\tau) - \frac{1}{2\tau}\int_0^{2\pi}\frac{d\sigma}{2\pi}(x(\sigma,\tau))^2 \end{aligned} \tag{4.2}$$

Invariance under constant shifts of σ is implemented by imposing [1]

$$\int d\sigma\left(\partial_\sigma x\partial_\tau x - \frac{1}{2\tau}\partial_\sigma x^2\right) = \int d\sigma\tau^2\partial_\sigma y\partial_\tau y = 0 \tag{4.3}$$

It is important that the two expressions for the lagrangian (4.1) and the expressions for the constraint (4.3) are invariant under the orbifold identification

$$\begin{aligned} &x(\sigma,\tau) \to x(\sigma,\tau) + 2\pi n\tau \\ &x_0^-(\tau) \to x_0^-(\tau) + 2\pi n\int_0^{2\pi}\frac{d\sigma}{2\pi}x(\sigma,\tau) + \frac{(2\pi n)^2}{2}\tau y(\sigma,\tau) \to y(\sigma,\tau) + 2\pi n \end{aligned} \tag{4.4}$$

The equations of motions for x_0^- and y_0^- set p^+ to a constant. The equations of motion for p^+ leads to

$$\begin{aligned} P_{x^-} &= p^+\partial_\tau x_0^- = \frac{1}{4\pi\alpha'}\int_0^\ell d\sigma\left(\partial_\tau x\partial_\tau x + \partial_\sigma x\partial_\sigma x\right) \\ P_{y^-} &= p^+\partial_\tau y_0^- = \frac{1}{4\pi\alpha'}\int_0^\ell d\sigma\tau^2\left(\partial_\tau y\partial_\tau y + \partial_\sigma y\partial_\sigma y\right) \end{aligned} \tag{4.5}$$

where we have rescaled σ to range in $[0, \ell = 2\pi\alpha' p^+)$ (we will continue to use this rescaled value). The hamiltonian P_{x^-} is not invariant under the orbifold identification (4.4) but P_{y^-} is invariant. Either expression can be used in the quantization [1].

A complete set of solutions to the equations of motion in the w-twisted sector, can be expressed in terms of harmonic oscillators:

$$\begin{aligned} x(\sigma,\tau) = \;&\xi + \frac{p}{p^+}\tau + \frac{2\pi w\sigma\tau}{\ell} + \\ &+ i\left(\frac{\alpha'}{2}\right)^{\frac{1}{2}}\sum_{n\neq 0}\left\{\frac{\alpha_n}{n}\exp\left[-\frac{2\pi i n(\sigma+\tau)}{\ell}\right] + \frac{\widetilde{\alpha}_n}{n}\exp\left[\frac{2\pi i n(\sigma-\tau)}{\ell}\right]\right\} \end{aligned} \tag{4.6}$$

The solution of x_0^- is obtained from (4.5). The solution for y is simply found from (4.2). Upon quantization these oscillators obey the standard canonical commutation relations.

The lagrangian (4.1) in terms of x is similar to the standard lagrangian and is expanded in terms of the normal modes in a way similar to the standard expansion. The differences in the zero modes are that $J = -\xi p^+$ must be quantized and the winding term $\frac{2\pi w\sigma\tau}{\ell}$ has an unusual form.

The evolution away from $\tau = 0$ can be analyzed using either the x or the y variables. The quantization in terms of x is standard. The coordinate ξ, which is not subject to identification on the singular surface $\tau = 0$ in $\mathcal{O}$ is constrained to be on the lattice $\xi =$

$-J/p^+$. The evolution of the nonzero modes is as in standard free string theory. The constraint (4.3) leads to $Jw + N - \widetilde{N} = 0$ where N and $\widetilde{N}$ are the standard number operators.

In terms of y variables we have a single-valued, but time dependent hamiltonian

$$P_y^- = \frac{1}{4\pi\alpha'}\int_0^\ell d\sigma \left[\frac{(2\pi\alpha'\Pi_y)^2}{\tau^2} + \tau^2(\partial_\sigma y)^2\right] = \frac{1}{2p^+}\left[\frac{J^2}{\tau^2} + \frac{w^2\tau^2}{\alpha'^2}\right] + \frac{\pi}{\ell}\sum_{n>0} H_n(\tau) \tag{4.7}$$

where $\Pi_y = \frac{1}{2\pi\alpha'}\tau^2\partial_\tau y$ is the canonical momentum and

$$\begin{aligned} H_n(\tau) &= \left[\lambda_n(\alpha_n^\dagger\alpha_n + \widetilde{\alpha}_n^\dagger\widetilde{\alpha}_n) + \rho_n\alpha_n\widetilde{\alpha}_n + \rho_n^*\alpha_n^\dagger\widetilde{\alpha}_n^\dagger + \omega_n\right] \\ \lambda_n &= 2 + \left(\frac{\ell}{2\pi n\tau}\right)^2 \\ \rho_n &= -\left(\frac{\ell}{2\pi n\tau}\right)^2\left[1 + \frac{4\pi i n\tau}{\ell}\right]\exp\left(-\frac{4\pi i n\tau}{\ell}\right) \\ \omega_n &= n\lambda_n \end{aligned} \tag{4.8}$$

The first term on the right hand of (4.7) has the interpretation of the "rotational kinetic energy" of the zeromode. The second term is the winding energy and the rest is the oscillator contribution.

Equation (4.7) has an interpretation in terms of the newtonian expression for the energy a particle with mass $\mu = p^+$ rotating around an origin at distance $r = \tau$ with angular momentum J:

$$H = \frac{J^2}{2\mu r^2} + V \tag{4.9}$$

Note once more the dual role of x^+ as both a radial and time coordinate. The Schrodinger problem with the hamiltonian (4.7) can be solved explicitly [1].

The term $\frac{\pi}{\ell}\sum_{n>0} n\lambda_n(t)$ in the hamiltonian (4.7) leads to a logarithmically divergent time dependent subtraction. It can be interpreted as a logarithmic subtraction in the definition of y_0^- in terms of x_0^- and the composite operator x^2 (4.2). Therefore, it is absorbed by a redefinition of the first term $-p^+\partial_\tau y_0^-$ in the lagrangian (4.1).

Since ∂_y is a Killing vector of (2.6) we can also consider the T-dual background with $g_{yy} \to 1/g_{yy}$ (see also [15, 18]). Define $\widetilde{y}$ by the T-duality transformation

$$\begin{aligned} 2\pi\Pi_y &= \frac{\tau^2\partial_\tau y}{\alpha'} = \partial_\sigma\widetilde{y} \\ 2\pi\Pi_{\widetilde{y}} &= \frac{\alpha'\partial_\tau\widetilde{y}}{\tau^2} = \partial_\sigma y \end{aligned} \tag{4.10}$$

where $\widetilde{y} \sim \widetilde{y} + 2\pi$. The dual action is

$$S' = \frac{1}{4\pi\alpha'}\int d\sigma d\tau\, \frac{(\alpha')^2}{\tau^2}\left[(\partial_\tau\widetilde{y})^2 - (\partial_\sigma\widetilde{y})^2\right] + \frac{1}{8\pi}\int d\sigma d\tau\,\sqrt{\gamma}\,R^{(2)}\log\frac{\alpha'}{(y^+)^2} \tag{4.11}$$

and the classical solution is

$$\widetilde{y} = \widetilde{y}_0 - \frac{\xi\sigma}{\alpha'} + \frac{2\pi w \tau^3}{3\alpha'\ell} + \tag{4.12}$$
$$+ p^+ \left(\frac{\alpha'}{2}\right)^{\frac{1}{2}} \sum_{n\neq 0} \left(1 + \frac{2\pi i n\tau}{\ell}\right) \left(\frac{\alpha_n}{n^2}\exp\left[-\frac{2\pi i n(\sigma+\tau)}{\ell}\right] - \frac{\widetilde{\alpha}_n}{n^2}\exp\left[\frac{2\pi i n(\sigma-\tau)}{\ell}\right]\right)$$

As expected, the winding number of $\widetilde{y}$, namely $-\frac{\ell\xi}{2\pi\alpha'} = -\xi p^+ = J$, is the eigenvalue of the parabolic generator J, while its momentum is $w = \int d\sigma\, \Pi_{\widetilde{y}}$, which was the winding number of y.

Let us now comment briefly on the evolution of the system when we include the singular subspace $\mathcal{F}_{x^+=0}$ or the singular subspace $\mathcal{F}'_{y^+=0}$. We have analyzed in detail the light-cone quantization of strings moving on both $\mathcal{O}$ and $\mathcal{O}'$. Depending on subtle points regarding the *interpretation* of singular gauge transformations and the geometry of $\mathcal{O}'$ it appears that string propagation on $\mathcal{O}$ differs from that on $\mathcal{O}'$. Moreover, a branelike object, which we call the "instabrane" might appear in the theory on $\mathcal{O}'$. The physical relevance of the instabrane is a subtle question which will be addressed in detail in [1].[5]

We remarked above that for an appropriate choice of spin structure on $\mathcal{O}-\mathcal{F}_0 \cong \mathcal{O}'-\mathcal{F}'_0$ the orbifold preserves some spacetime supersymmetries. It is straightforward to extend the worldsheet lagrangian (4.1) to the Green-Schwarz formalism. For concreteness consider the model on $\mathcal{O}\times\mathbb{R}^7$. Before taking the quotient by Γ we should add to (4.1) seven free worldsheet bosons x^i and eight rightmoving fermions S^a (in the type II theory we also need eight leftmoving fermions and in the heterotic string also leftmoving degrees of freedom for the internal degrees of freedom). It is easy to see using the symmetries of the problem that after the action by Γ the added fields x^i and S^a remain free and periodic around the string [1].[6]

5. Covariant formulation

5.1 Quantization and exchange algebra

The covariant action is

$$S = \frac{1}{4\pi\alpha'}\int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma\ \eta_{\mu\nu}\left(\partial_\tau x^\mu \partial_\tau x^\nu - \partial_\sigma x^\mu \partial_\sigma x^\nu\right) \tag{5.1}$$

In the twisted sector the boundary conditions are, $X(\sigma+2\pi,\tau) = e^{2\pi w\mathcal{J}}X(\sigma,\tau)$ where $w \in \mathbf{Z}$ ($\mathcal{J}$ was defined in (2.1)). The general solution to the equations of motion in the twisted sector may be expressed in terms of oscillators

$$\begin{aligned} \widehat{x}^\mu_L &:= i\sum_{n\neq 0}\frac{\alpha^\mu_n}{n}e^{-inu^+} \\ \widehat{x}^\mu_R &:= i\sum_{n\neq 0}\frac{\widetilde{\alpha}^\mu_n}{n}e^{inu^-} \end{aligned} \tag{5.2}$$

[5] If instabranes are indeed not gauge equivalent to Fock space states then a number of interesting questions arise. For examples: Can they probe the geometry near the singularity? Are they associated with K-theory of the crossed-product $C(\mathbb{R}^{1,2}) \ltimes \Gamma$?

[6] The same conclusion was also reached by A. Tseytlin.

where $u^{\pm} := \sigma \pm \tau$, and the zeromodes,

$$X_z(\tau) := \begin{pmatrix} x_0^+ + \alpha' p^+ \tau \\ x_0 + \alpha' p \tau \\ x_0^- + \alpha' p^- \tau + w^2 \left(\alpha' p^+ \frac{\tau^3}{6} + x_0^+ \frac{\tau^2}{2}\right) \end{pmatrix} \tag{5.3}$$

by introducing a "spectral flow" operator:

$$X(\sigma,\tau) = \exp(w\sigma\mathcal{J})X_z(\tau) + \exp(wu^+\mathcal{J})\widehat{X}_L(u^+) + \exp(wu^-\mathcal{J})\widehat{X}_R(u^-). \tag{5.4}$$

The unusual nature of the solution (5.4) leads to novel commutation relations on the oscillators. The symplectic form is standard, $\Omega = \frac{1}{2\pi\alpha'}\int d\sigma\ \delta x^\mu \eta_{\mu\nu}\partial_\tau \delta x^\nu$, but the canonical commutation relations of the oscillators become:

$$\begin{aligned} [\alpha_n^\mu, \alpha_m^\nu] &= n\delta_{n+m,0}\left[\frac{1}{\eta + i\frac{w}{n}\eta\mathcal{J}}\right]^{\mu\nu} \\ [\widetilde{\alpha}_n^\mu, \widetilde{\alpha}_m^\nu] &= n\delta_{n+m,0}\left[\frac{1}{\eta - i\frac{w}{n}\eta\mathcal{J}}\right]^{\mu\nu} \end{aligned} \tag{5.5}$$

while $[\alpha, \widetilde{\alpha}] = 0$.

Equations (5.5) lead to a curious exchange algebra. One way to express this is to introduce single-valued fields $J_\pm(u^\pm) := e^{-wu^\pm\mathcal{J}}\partial_\pm X$ which satisfy commutation relations

$$\begin{aligned} \left[j_+^\mu(u^+), j_+^\nu(u^{+'})\right] &= \pi i\alpha'\eta^{\mu\nu}\partial_+\delta^{(p)}(u^+ - u^{+'}) + \pi i\alpha'\delta^{(p)}(u^+ - u^{+'})F^{\mu\nu} \\ \left[j_-^\mu(u^-), j_-^\nu(u^{-'})\right] &= -\pi i\alpha'\eta^{\mu\nu}\partial_-\delta^{(p)}(u^- - u^{-'}) - \pi i\alpha'\delta^{(p)}(u^+ - u^{+'})F^{\mu\nu} \\ F := w\mathcal{J}\eta^{-1} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -w \\ 0 & w & 0 \end{pmatrix} \end{aligned} \tag{5.6}$$

Unlike $\partial_\pm X$, the $J_\pm$ are not conformal fields. If we rotate to euclidean signature and consider radial evolution in the complex plane, then (5.6) is equivalent to an exchange algebra in the w-twisted sector. For $|z_1| > |z_2|$ we have

$$\partial x^{\mu_1}(z_1)\partial x^{\mu_2}(z_2) = \partial x^{\mu_2}(z_2)\partial x^{\mu_1}(z_1) + i\frac{w}{z_1 z_2}\left(e^{-iw\log z_1\mathcal{J}}\mathcal{J}e^{iw\log z_2\mathcal{J}}\eta^{-1}\right)^{\mu_1\mu_2}. \tag{5.7}$$

The exchange algebra (5.7) is very similar to the exchange algebras of chiral vertex operators of RCFT, and indeed the above analysis applies to *any* orbifold by a *linear* action on the spacetime coordinates. Note that the exchange algebra has the flavor of an Heisenberg algebra, suggestive of non-commuting coordinates and hence of non-commutative geometry. Since it is present only in the twisted sectors with $w \neq 0$, and the wound strings are light only near $x^+ = 0$, it is reasonable to think that it reflects a property of the region near $\mathcal{F}_{x^+=0}$. In the context of another time-dependent orbifold, Nekrasov [6] has advocated a role of quantum groups. He discusses a two-dimensional model, and considers D0 branes, while we are working in the chiral sector of the closed string. The two results are different, but they are similar in spirit. Possible noncommutativity in light-cone coordinates was also discussed in [9].

5.2 The physical state conditions

Although the worldsheet energy-momentum tensor has a standard form in X-coordinates:

$$T_{++} = \frac{1}{\alpha'}\partial_+ X^{tr}\eta\partial_+ X = \sum_{n\in Z} L_n e^{in(\tau+\sigma)} \tag{5.8}$$

we have a nonstandard realization of Virasoro operators since $\mathcal{J}$ is not diagonalizable, making the solution of the physical state conditions nontrivial. Nevertheless, a DDF-operator construction shows the BRST cohomology is nontrivial in the twisted sectors. More explicitly, it is easy to construct physical states in the winiding sector with zero modes only

$$\sqrt{\frac{p^+}{ix_0^+}}\exp\left[-ip^+x_0^- - i\frac{\mathrm{m}^2}{2p^+}x_0^+ + i\frac{p^+}{2x_0^+}(x_0+\frac{J}{p^+})^2 - i\frac{w^2(x_0^+)^3}{6(\alpha')^2p^+}\right] \tag{5.9}$$

where $\mathrm{m}^2 := m^2 + \vec{p}_\perp^{\,2}$, while $\vec{p}_\perp$ is the momentum in the transverse directions and $m^2 = -4/\alpha'$. One then can show that DDF-like operators

$$A_n = \oint \frac{dz}{2\pi}\left(\frac{2}{\alpha'}\right)^{\frac{1}{2}}\left[\partial_z x + iw\log z\partial_z x^+ + \frac{w}{znk_0}\right]e^{ink_0x^+}(z)\,, \qquad n = \pm1, \pm2, \dots \tag{5.10}$$

with $k_0 = \frac{2}{\alpha' p^+}$, are well-defined and survive the orbifold projection. Together with their rightmoving counterparts $\widetilde{A}_n$, they act on (5.9) to generate physical states with a full tower of positive signature Fock space states, even in the twisted sectors. This will be confirmed by the computation of the trace in covariant quantization.

5.3 Torus partition function

Let us now consider the path integral on a torus with worldsheet metric

$$g = dz_+dz_- = (d\sigma^1 + \tau_+ d\sigma^2)(d\sigma^1 + \tau_- d\sigma^2) \tag{5.11}$$

where σ^1, σ^2 have period 1, while $\tau_\pm \in \mathbb{R}$ for lorentzian signature, and $\tau_+ = (\tau_-)^*$ for euclidean signature tori. Until further notice we will use a lorentzian torus.

In the orbifold theory we sum over winding sectors (w_a, w_b) around the a,b-cycles. In these sectors we write the field as

$$X(\sigma^1,\sigma^2) = \exp\left[2\pi(\sigma^1 w_a + \sigma^2 w_b)\mathcal{J}\right]\sum_{n_a,n_b\in\mathbf{Z}} X_{n_a,n_b}e^{2\pi i(n_a\sigma^1+n_b\sigma^2)}\,. \tag{5.12}$$

Because the matrix $\mathcal{J}$ is strictly lower triangular the contribution of the quantum fluctuations is *independent* of w_a, w_b, and one finds that the one-loop partition function for the string theory is:

$$Z = \int_{\mathcal{O}} \frac{d^3x}{(2\pi\sqrt{\alpha'})^3}\times \tag{5.13}$$
$$\times\sum_{w_a,w_b\in\mathbf{Z}}\exp\left[-i\pi\frac{(x^+)^2}{\alpha'}\frac{(w_b+w_a\tau_+)(w_b+w_a\tau_-)}{\tau_2}\right]\frac{-iZ^{\mathrm{ghost}}Z^\perp(w_a,w_b)}{(-i\tau_2)^{3/2}(\eta(\tau_+)\eta(-\tau_-))^3}$$

with $\tau_2 = (\tau_+ - \tau_-)/2$. This formula can also be derived by explicit evaluation of the trace

$$Z^{\text{matter}} = \operatorname*{Tr}_{\mathcal{H}} e^{2\pi i \tau_+ (L_0 - c/24)} e^{-2\pi i \tau_- (\widetilde{L}_0 - c/24)} \tag{5.14}$$

where $\mathcal{H}$ is the CFT state space of the orbifold. In this second derivation it is important to take $\tau_\pm$ real since L_0 is not bounded below.

Of course, (5.13) bears a striking resemblance to the partition function for the partition function for a gaussian field with target space a circle of radius x^+. Once again, the dual role of x^+ as both a radial and a time variable leads to some interesting physics.

Note that we have left the integration over the zeromode $x^\mu = X^\mu_{00}$ of the field X undone in (5.13).[7] This is important for the correct conceptual interpretation of the amplitude: We have a spacetime-dependent contribution to the cosmological constant

$$\begin{aligned}\Lambda(x^\mu) &= \frac{i}{(2\pi\sqrt{\alpha'})^3} \sum_{w_a, w_b \in \mathbf{Z}} \int_{\mathcal{F}} \frac{d\tau_+ \wedge d\tau_-}{(\tau_2)^2} \exp\left[-i\pi \frac{(x^+)^2}{\alpha'} \frac{(w_b + w_a \tau_+)(w_b + w_a \tau_-)}{\tau_2}\right] \times \\ &\times \frac{Z^\perp(w_a, w_b)}{(-i\tau_2)^{1/2}(\eta(\tau_+)\eta(-\tau_-))}\,. \end{aligned} \tag{5.15}$$

At this point, we will take the torus to have euclidean signature. Some of the issues that arise when attempting to make sense of the lorentzian signature formulae were discussed in [19] and will be further addressed in [1].

The expression (5.15) exhibits a curious divergence as $x^+ \to 0$. Suppose, for simplicity, that Γ does not act on $\mathcal{C}^\perp$. Then we may also write:

$$\Lambda(x^+) = -\frac{1}{(2\pi)^3 \alpha' |x^+|} \int_{\mathcal{F}} \frac{d^2\tau}{(\tau_2)^2} \frac{Z^\perp}{|\eta(\tau)|^2} \sum_{w_a, \widehat{w}_b \in \mathbf{Z}} q^{\frac{1}{2}\alpha' p_L^2} \bar{q}^{\frac{1}{2}\alpha' p_R^2} \tag{5.16}$$

with $p_L = \frac{1}{\sqrt{2}}(\frac{\widehat{w}_b}{x^+} + w_a \frac{x^+}{\alpha'})$ and $p_R = \frac{1}{\sqrt{2}}(\frac{\widehat{w}_b}{x^+} - w_a \frac{x^+}{\alpha'})$. The modes with $\widehat{w}_b = 0, w_a \neq 0$ are *light winding modes* for $x^+ \to 0$. This shows there is a *divergence* as $x^+ \to 0$:

$$\Lambda(x^+) \sim \frac{1}{(x^+)^2} \times \int_{\mathcal{F}} \frac{d^2\tau}{(\tau_2)^2} \frac{Z^{\text{tr}}}{|\eta(\tau)|^2} \tag{5.17}$$

This divergence has a simple interpretation as a volume divergence in the T-dual coordinate $\widetilde{y}$ of radius $1/x^+$. This suggests that the light winding strings "open up" the singularity from a cone to a trumpet. Of course, we cannot conclude this is the correct qualitative physics until we have carefully considered one-loop interactions and issues of backreaction.

The above formulae apply to the superstring in the NSR formalism. In this case $Z^\perp$ includes the contributions of the NSR fermions. When we choose the supersymmetric spin structure on the space time fermions the phases in the sum over the spin structures and the partition functions themselves are independent of (w_a, w_b), and hence (5.17) multiplies zero. If we choose the other spin structure in spacetime, the situation is similar to that in [20]. Then there are tachyons for $|x^+|$ smaller than the string scale and a negative cosmological constant is generated.

[7] Of course, we could perform the x^+ integral. In this case we obtain a nonholomorphic Eisenstein series.

6. Interactions

6.1 Tree level interactions

We have studied tree level interactions in ϕ^3 field theory on the orbifold, in string theory in light-cone gauge, and in the covariant formulation. Here we briefly summarize some results about tree level amplitudes deferring more details to [1]. The main result is that for generic momenta the amplitudes are finite, but when the intermediate particles have $p^+ = 0$ the amplitudes can diverge. These divergences are associated with the singularity at $x^+ = 0$.

At large $|x^+|$, the geodesic distances between the image points go to infinity; i.e. the space effectively opens up. Thus we expect to be able to prepare our "in" and "out" states in the far past and far future, and to consider the S-matrix elements similar to those in $\mathbb{R}^{1,2}$. The standard S-matrix elements in $\mathbb{R}^{1,2}$ are expressed in the plane wave basis (3.4). However, the plane waves are not invariant under the orbifold action and the invariant functions constructed from them by summing over the images are not convenient to work with. A better basis is the J basis $\psi_{p^+,J}$ of (3.3), in terms of which the orbifold projection simply corresponds to taking J to be integral. This motivates us to consider the S-matrix elements in flat $\mathbb{R}^{1,2}$ in the J basis. This is somewhat unusual since going to the J-basis introduces time dependence which arises because J and p^- do not commute. More explicitly, the wave function $\psi_{p^+,J}$ of a free particle is forced to be localized at $\xi = -J/p^+$ at $x^+ = 0$, which breaks the translational invariance of x^+. In other words, the on-shell "in" and "out" states are prepared in a way that they are aimed at points $\xi_i = -J_i/p^+$ (i labels particles) so that at $x^+ = 0$, their wavepackets are completely localized at the respective points. In flat $\mathbb{R}^{1,2}$, this corresponds to asking time-dependent questions in a time independent background. S-matrix elements in the J-basis are observables in $\mathbb{R}^{1,2}$ which exhibit some features of time dependent amplitudes.

Since $\psi_{p^+,J}$ is obtained from the plane waves by a Fourier transform (3.3), the amplitudes in the J basis can be obtained from those in the standard momentum basis by a Fourier transform. Once these tree level S-matrix elements are computed in flat space, it is straightforward to compute them for the orbifold simply by restricting J_i to be integers.

In the following we give the results for the three-point and four-point tree-level amplitudes for untwisted tachyons on the orbifold. More details will appear in [1].

From (3.3) the vertex operator for the tachyon in the J-basis can be written as

$$V_{p_i^+,J_i}(\sigma,\tau) = \frac{1}{\sqrt{2\pi p_i^+}} \int_{-\infty}^{\infty} dp\, e^{ip\xi_i}\, e^{i\vec{p}_i\cdot\vec{X}(\sigma,\tau)}, \qquad \xi_i = -\frac{J_i}{p_i^+} \tag{6.1}$$

where i labels external particles, the factor of $\frac{1}{\sqrt{p_i^+}}$ is introduced for the natural normalization of states in the light-cone frame, and

$$e^{i\vec{p}_i\cdot\vec{X}(\sigma,\tau)} = \exp\left[-ip_i^+x^- - ip_i^-x^+ + ip_ix + i\vec{p}_{\perp i}\cdot\vec{x}_\perp(\sigma,\tau)\right] \tag{6.2}$$

($\vec{p}_{\perp i}$ and $\vec{x}_{\perp i}$ denote vectors in the transverse dimensions) is the standard on-shell tachyon vertex operator with

$$p_i^- = \frac{p_i^2 + \mathrm{m}_i^2}{2p_i^+}\,, \qquad \mathrm{m}_i^2 = m^2 + \vec{p}_{\perp i}^{\,2}\,, \qquad m^2 = -\frac{4}{\alpha'}\,. \tag{6.3}$$

Three point function. The $1 \to 2+3$ tachyon amplitude in the J-basis in the covering space is [1]

$$A_3 = \frac{8\pi i g_s}{\alpha'}(2\pi)^{25}\,\delta(p_1^+ - p_2^+ - p_3^+)\,\delta(\vec{p}_{\perp 1} - \vec{p}_{\perp 2} - \vec{p}_{\perp 3})\,\delta(J_1 - J_2 - J_3)\,w_3(J_i, p_i^+) \tag{6.4}$$

with

$$\begin{aligned} w_3(p_i^+; J_i) &= \int_{-\infty}^{\infty} dx^+\, \frac{1}{(-ix^+)^{\frac{1}{2}}}\, e^{-\frac{i}{2}\alpha(p_i^+,\mathrm{m}_i)x^+ - \frac{i}{2x^+}\mu_{23}(\xi_2-\xi_3)^2} \\ &= \begin{cases} 2\sqrt{\frac{2\pi}{\alpha}}\cos\left(\sqrt{\alpha\mu_{23}}(\xi_2 - \xi_3)\right), & \alpha > 0 \\ 0, & \alpha < 0 \end{cases}. \end{aligned} \tag{6.5}$$

In (6.5) we have defined

$$\begin{aligned} \alpha(p_i^+, \mathrm{m}_i) &= \frac{\mathrm{m}_1^2}{p_1^+} - \frac{\mathrm{m}_2^2}{p_2^+} - \frac{\mathrm{m}_3^2}{p_3^+}\mu_{23} \\ &= \frac{p_2^+ p_3^+}{p_2^+ + p_3^+} \end{aligned} \tag{6.6}$$

Going to the orbifold, we take J_i to be integers and replace the Dirac delta function for J_i by δ_{J_1, J_2+J_3} as a result of dividing the amplitude by the volume of the orbifold group.

It is worth noting that the amplitude is not an analytic function of α. Such lack of analyticity in the "momenta" is common in problems with a noncompact inhomogeneous target space such as the $c = 1$ system [21]–[25]. The novelty here is that the inhomogeneous direction is not spacelike but lightlike.

These amplitudes have a simple physical interpretation in the light-cone description of the theory. We have already mentioned that in the light-cone description we interpret the system as nonrelativistic particles with time, mass and potential energy

$$\tau = x^+, \quad \mu_i = p_i^+\,, \qquad V_i = \mathrm{m}_i^2/2\mu_i \tag{6.7}$$

In this notation the functions $\psi_{p^+,J} = \sqrt{\frac{\mu}{i\tau}}e^{-i\mu x^- - iV\tau + i\frac{\mu}{2\tau}(x-\xi)^2}$ are the nonrelativistic propagator from a position x at time τ to the position $\xi = -J/p^+$ at time $\tau = 0$ times the trivial factor $e^{-i\mu x^- - iV\tau}$. This is another way to understand the focusing at $x = \xi$ at $x^+ = 0$.

In terms of the notation (6.7) the decay process is described as a particle with mass μ_1 decaying to two particles with with masses μ_2 and μ_3. p^+ conservation, which arises from translational invariance of x^-, is being interpreted as conservation of mass

$$\mu_1 = \mu_2 + \mu_3\,. \tag{6.8}$$

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The center of mass coordinate of the decay products is $\frac{\mu_2 x_2+\mu_3 x_3}{\mu_2+\mu_3}$. J conservation guarantees that its value at $x^+ = 0$ is the same as the value of x_1 at $x^+ = 0$

$$\xi_1 = \frac{\mu_2\xi_2 + \mu_3\xi_3}{\mu_2 + \mu_3} \tag{6.9}$$

The remaining dynamics is best described in terms of the relative coordinate and the reduced mass

$$x_{23} = x_2 - x_3\,, \qquad \mu_{23} = \frac{\mu_2\mu_3}{\mu_2 + \mu_3} \tag{6.10}$$

The parameter α defined in (6.6) takes the form

$$\alpha = 2(V_1 - V_2 - V_3) \tag{6.11}$$

and is interpreted as the difference in potential energy between particle 1 and particles 2 and 3. An obvious necessary kinematical condition for the decay is $\alpha = 2(V_1-V_2-V_3) > 0$, as is the case in (6.5). This also explains why the amplitude is nonanalytic as a function of α.

The decay amplitude (6.5), can be rewritten as

$$w_3(\mu_i,\xi_i) = \int_{-\infty}^{\infty} dx^+ \frac{1}{\sqrt{-ix^+}} \exp\left[i(V_2+V_3-V_1)x^+ - i\frac{\mu_{23}(\xi_2-\xi_3)^2}{2x^+}\right]. \tag{6.12}$$

We see an integral over the time of the interaction x^+. The integrand has two phases. The first term, as discussed above, is associated with the difference in potential energy. The second term, is the free propagation of the relative coordinate of the decay products from its value $x_{23} = \xi_2 - \xi_3$ at $x^+ = 0$ to its value $x_{23} = 0$ at the time of the interaction x^+.

Four point function. The four point function probes the structure of the interacting theory in more detail than the three point function. As a particle approaches the singularity it is blue shifted, its energy becomes large, and its coupling to the graviton becomes large. Therefore, one expects large back reaction of the geometry which could appear as a divergence in the four point function. The same argument can be used for the correlation functions of (6.1) in the covering space because the focusing at ξ_i leads to large energy density and potentially large back reaction.

A signal of such a problem would be the failure of the convergence of the Fourier transform of the Virasoro-Shapiro amplitude. We therefore study this transform in some detail.

The Fourier transform of the Virasoro-Shapiro amplitude can be written as

$$A_4 = \frac{8(2\pi)^3 i g_s^2}{\alpha'} \int \left(\prod_{i=1}^{4} \frac{dp_i}{\sqrt{2\pi p_i^+}}\right) \delta(p_1+p_2-p_3-p_4) e^{iF}\delta(E)A(s,t) \tag{6.13}$$

where we suppressed a factor of $(2\pi)^{24}\delta(p_1^+ + p_2^+ - p_3^+ - p_4^+)\delta(\vec{p}_{\perp 1} + \vec{p}_{\perp 2} - \vec{p}_{\perp 3} - \vec{p}_{\perp 4})$ and

$$\begin{aligned} A(L_s, L_t, L_u) &= \pi \frac{\Gamma\left(-\frac{\alpha'}{4}L_s\right)\Gamma\left(-\frac{\alpha'}{4}L_t\right)\Gamma\left(-\frac{\alpha'}{4}L_u\right)}{\Gamma\left(1+\frac{\alpha'}{4}L_s\right)\Gamma\left(1+\frac{\alpha'}{4}L_t\right)\Gamma\left(1+\frac{\alpha'}{4}L_u\right)} \\ &= -\left(\frac{\Gamma\left(-\frac{\alpha'}{4}L_t\right)\Gamma\left(-\frac{\alpha'}{4}L_u\right)}{\Gamma\left(1+\frac{\alpha'}{4}L_s\right)}\right)^2 \frac{\sin\left(\frac{\alpha'\pi}{4}L_t\right)\sin\left(\frac{\alpha'\pi}{4}L_u\right)}{\sin\left(\frac{\alpha'\pi}{4}L_s\right)} \end{aligned} \tag{6.14}$$

and

$$\begin{aligned} E &= p_1^- + p_2^- - p_3^- - p_4^- = \frac{p_1^2 + \mathrm{m}_1^2}{2p_1^+} + \frac{p_2^2 + \mathrm{m}_2^2}{2p_2^+} - \frac{p_3^2 + \mathrm{m}_3^2}{2p_3^+} - \frac{p_4^2 + \mathrm{m}_4^2}{2p_4^+} \qquad (6.15) \\ F &= p_1\xi_1 + p_2\xi_2 - p_3\xi_3 - p_4\xi_4 \\ L_s &= s - m^2 + i\epsilon = 2(p_1^+ + p_2^+)(p_1^- + p_2^-) - (p_1 + p_2)^2 - \mathrm{m}_s^2 + i\epsilon \\ L_t &= t - m^2 + i\epsilon = 2(p_1^+ - p_3^+)(p_1^- - p_3^-) - (p_1 - p_3)^2 - \mathrm{m}_t^2 + i\epsilon \\ L_u &= u - m^2 + i\epsilon = 2(p_1^+ - p_4^+)(p_1^- - p_4^-) - (p_1 - p_4)^2 - \mathrm{m}_u^2 + i\epsilon \\ \mathrm{m}_s^2 &= m^2 + (\vec{p}_{\perp 1} + \vec{p}_{\perp 2})^2, \qquad \mathrm{m}_t^2 = m^2 + (\vec{p}_{\perp 1} - \vec{p}_{\perp 3})^2, \qquad \mathrm{m}_u^2 = m^2 + (\vec{p}_{\perp 2} - \vec{p}_{\perp 3})^2 \end{aligned}$$

The momentum integrals can be reduced to a single integral expression

$$A_4 = \frac{8(2\pi)^2 i g_s^2}{\alpha'}\delta(J_1 + J_2 - J_3 - J_4)\int_{-\infty}^{\infty}\frac{dq}{|q|}\exp\left[\frac{i}{2}\left(q\xi_- + \frac{\alpha\xi_+}{q}\right)\right] A(L_s, L_t, L_u) \tag{6.16}$$

with

$$\begin{aligned} L_s &= (p_1^+ + p_2^+)\left(q_+^2 + \frac{\mathrm{m}_1^2}{p_1^+} + \frac{\mathrm{m}_2^2}{p_2^+}\right) - \mathrm{m}_s^2 + i\epsilon \\ L_t &= (p_3^+ - p_1^+)\left(\frac{\mathrm{m}_3^2}{p_3^+} - \frac{\mathrm{m}_1^2}{p_1^+}\right) - \mathrm{m}_t^2 - \mu_{12}\left(\sqrt{\frac{p_3^+}{p_1^+}}q_+ - \sqrt{\frac{p_4^+}{p_2^+}}q_-\right)^2 + i\epsilon \\ L_u &= (p_4^+ - p_1^+)\left(\frac{\mathrm{m}_4^2}{p_4^+} - \frac{\mathrm{m}_1^2}{p_1^+}\right) - \mathrm{m}_u^2 - \mu_{12}\left(\sqrt{\frac{p_4^+}{p_1^+}}q_+ + \sqrt{\frac{p_3^+}{p_2^+}}q_-\right)^2 + i\epsilon \\ \xi_\pm &= \sqrt{\mu_{12}}(\xi_1 - \xi_2) \pm \sqrt{\mu_{34}}(\xi_3 - \xi_4) \\ q_\pm &= \frac{1}{2}\left(q \pm \frac{\alpha}{q}\right) \\ \mu_{12} &= \frac{p_1^+ p_2^+}{p_1^+ + p_2^+}, \qquad \mu_{34} = \frac{p_3^+ p_4^+}{p_3^+ + p_4^+} \\ \alpha &= \frac{\mathrm{m}_3^2}{p_3^+} + \frac{\mathrm{m}_4^2}{p_4^+} - \frac{\mathrm{m}_1^2}{p_1^+} - \frac{\mathrm{m}_2^2}{p_2^+} \end{aligned} \tag{6.17}$$

This is an integral over various values of s with corresponding values of t and u. The integral passes through the poles in the s-channel. These do not lead to divergences because of the $i\epsilon$. Instead, these poles make the amplitude nonanalytic as a function of the external momenta $\vec{p}_i$. This nonanalyticity is similar to the nonanalyticity we have already observed as a function of α in the three point function and its origin is similar. It is associated with large $|x^+|$.

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Let us analyze the amplitude in more detail and consider first the situation of generic p_i^+. Possible divergences and nonanalyticity in ξ_i can arise only from the behavior of the integral (6.16) at $q = 0, \pm\infty$, which correspond to the interesting region $x^+ \approx 0$. For generic p_i^+ each of these limits corresponds to the hard scattering limit of large s, t, u with fixed ratios. In this limit $A(L_s, L_t, L_u)$ decays rapidly and the integral (6.16) converges for all ξ_i. The dependence on ξ_i is analytic.

We would like to contrast this result with the corresponding behavior in field theory. Because the large momentum dependence of the field theory amplitude is power like, the Fourier transforms do not share the nice analytic structure in ξ that the string amplitudes enjoy. We conclude that the better UV behavior of string theory improves the analytic structure of the amplitudes as a function of ξ_i.

For nongenric p_i^+ the situation is different. Consider the kinematical configuration with vanishing $p_t^+ = p_3^+ - p_1^+ = p_2^+ - p_4^+$. Then the large q behavior of (6.16) is in the Regge region and $A(L_s, L_t, L_u) \sim q^{-\alpha' \mathrm{m}_t^2} = q^{4-\alpha' \vec{p}_{\perp t}^2}$ with $\vec{p}_{\perp t} = \vec{p}_{\perp 3} - \vec{p}_{\perp 1}$. When the exchanged particle has vanishing $J_t = J_3 - J_1 = 0$, i.e. $\xi_1 = \xi_3$, and $\xi_2 = \xi_4$, the integral over q diverges[8] for $\alpha' \vec{p}_{\perp t}^2 < 4$. To see the divergence in more detail consider the scattering with $J_t = 0$ and $p_t^+ = p_3^+ - p_1^+ \to 0$. Then the amplitude behaves as $A_4 \sim (p_t^+)^{-4+\alpha' \vec{p}_{\perp t}^2}$, up to logarithmic corrections. We interpret this behavior which arises from the Regge region as due to the exchange in the t-channel of particles with "spin" $\alpha(t) = 2 - \frac{1}{2}\alpha' \vec{p}_{\perp t}^2$. For $\vec{p}_{\perp t} = 0$ they include the graviton. By exchanging $3 \leftrightarrow 4$ we find a similar situation in the u-channel.

We conclude that even though the amplitude for fixed generic momenta is finite, the total cross section diverges due to singularities associated with particles with $p^+ = 0$ in the t and the u channel. A more detailed analysis of the amplitude shows that these particles are exchanged at $x^+ \approx 0$, and therefore can be interpreted as associated with the singularity and the large energy density there. We do not fully understand the implication of this phenomenon and it might be a signal of the breakdown of perturbation theory.

It is important to point out that the amplitude we have just derived cannot be obtained by a straightforward application of the formalism based on euclidean worldsheets. Since the target space has lorentzian signature, we expect the worldsheet to have lorentzian signature. Furthermore, a Fourier transform similar to (6.13) of the standard density on the euclidean signature moduli space need not converge. On the other hand, an expression for the amplitude with a lorentzian worldsheet appears to exist, and to have a nonrelativistic interpretation similar to that of the three point function.

6.2 Tadpoles and backreaction

We have remarked above on the question of backreaction of the geometry to incoming particles. Another important question is whether our background is stable against quantum fluctuations. Since it does not have closed timelike curves, we do not expect Hawking's chronology conjecture [26] to demand the existence of new singularities. However, the

[8] When $\xi_1 \neq \xi_3$, the integral converges. Physically, the convergence for $\xi_1 \neq \xi_3$ arises since the external particles are not focused at the same point.

presence of closed null curves at $x^+ = 0$ is potentially dangerous. In standard orbifolds the expectation values of composite operators like the stress tensor diverge at fixed points. In lorentzian orbifolds such divergences occur along closed null curves (see e.g. [27, 28], and references therein). For example, the expectation value of the stress tensor of a free scalar field in the similar problem of a BTZ black hole was calculated in [29]–[31], and for fermions in [32] (for a review see [33]). Such an expectation value acts as a source for the gravitational field and destabilizes the background.

We have already seen that in the background with the nonsupersymmetric spin structure a cosmological constant is being generated. It is therefore clear that this background is unstable. A cosmological constant is not generated at one loop in the supersymmetric model, and therefore it makes sense to examine this background in more detail. More specifically we consider $\mathbb{R}^7 \times \mathcal{O}$. Using the symmetries of our problem (supersymmetry and the two Killing vectors ∂_y and ∂_{y^-}) and the results of [34] it is straightforward to show that the most general metric and dilaton field are

$$\begin{aligned} (ds)^2 &= -2dy^+dy^- + \left(f\left(y^+\right)\right)^2 (dy)^2 + \left(dx^i\right)^2 \\ D &= D(y^+) \end{aligned} \tag{6.18}$$

At tree level $f(y^+) = y^+$ and D is a constant. This raises the question of whether they have radiative corrections. We have examined the one-loop tadpoles for on shell physical particles, and found them to vanish. The only possible tadpoles come from the graviton with polarization along the orbifold or the dilaton. In the tadpole computation we examined the torus expectation values of these vertex operators with arbitrary p^-, since the background depends on x^+. In the covariant formulation the expectation values turn out to vanish because of the sum over the spin structures. In the light-cone Green-Schwarz formalism they vanish because of the presence of fermion zero modes. Details (and subtleties) of these computations will be discussed in [1].

We conclude that up to one loop order the zero and one point functions vanish and our background is stable. However, the subtleties with intermediate particles with $p^+ = 0$ we encountered in the tree level four point function prevent us from concluding that the background is absolutely stable. Such particles can appear in the two point function at one loop or the zero point function at two loops and could potentially destabilize our orbifold.

7. Conclusions and future directions

We have studied the parabolic orbifold following the standard rules of perturbative string theory. We have found that the parabolic orbifold makes suprisingly good sense, despite the preponderance of potentially pathological pitfalls. One cloud on the horizon is the divergence of the 4 point function in special kinematic configurations. This needs to be understood much more thoroughly.

The issue of backreaction is currently under study, and might prove to be a serious problem with future development of this example. One way backreaction could ruin the orbifold is through the coupling of gravity to the large energy momentum of particles which are blue shifted near the singularity. Indeed, we found that although the tree level four point function is finite for generic momenta, it diverges when the exchanged particle in the t or u channel has vanishing p^+. This could signal the breakdown of perturbation theory. An interesting possibility, which has been suggested by numerous physicists, is that every scattering process might create a black hole.

In the spirit of black hole complementarity [35] it might be that the system admits two complementary descriptions. One of them, for an observer at $x^+ < 0$ is in terms of a Universe which ends at the singularity. The other description includes both sides of the space on the two sides of the singularity.

We have found that there are tantalizing hints of a role of noncommutative geometry in resolving the $x^+ = 0$ singularity, and in making sense of string theory on the non-Hausdorff quotient space. Perhaps related to this is the nontrivial exchange algebra for the coordinate currents ∂x^μ in the twisted sectors, a result which is possibly related in turn to that of [6].

We believe there are many other interesting examples of noncompact orbifolds associated with noncompact discrete groups. The flavor of these models is rather different from that of the compact orbifolds by finite groups which have been much studied in the context of Calabi-Yau compactification. This new territory might hold some exciting new surprises.

Note added: After this paper was submitted it was suggested by several people and was shown more decisively in [36]–[38] that the singularity in the four point function indeed reflects a large backreaction of the geometry to incoming particles. This backreaction renders perturbation theory invalid. However, the techniques presented here are useful in analyzing closely related models which are not singular.

Acknowledgments

We thank B. Acharya and A. Tseytlin for participating in the early stages of this project and discussions. We also thank T. Banks, S. Frolov, J. Harvey, G. Horowitz, D. Kutasov, J. Maldacena, E. Martinec, A. Strominger, C. Thorn and E. Witten for very helpful discussions. HL and GM were supported in part by DOE grant #DE-FG02-96ER40949 to Rutgers. NS was supported in part by DOE grant #DE-FG02-90ER40542 to IAS. GM would like to thank the Isaac Newton Institute for hospitality during the completion of this manuscript.

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